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Convergence of Sequences of Linear Positive Operators: Remarks and Applications

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1. INTRODUCTION

Korovkin's well-known result, [4, p. 14] and [5, p. 7], states that for a sequence L_n of positive linear operators and for a continuous function f, $L_n(f; x)$ converges uniformly to f provided $L_n(f_i, x)$ converges uniformly to $f_i(x)$ $(i = 0, 1, 2)$, f_i being appropriate functions. Shisha and Mond, in [6] and [7], and DeVore [1] determined the rate of convergence of such sequences $L_n(f, x)$ in terms of the moduli of continuity of f and f'.

Our purpose is to modify these results on the rate of convergence so as to apply to operators defined for functions on, e.g., $[0, \infty)$ or $(-\infty, \infty)$. Applications to Baskakov and Szász operators and various convolution operators are given yielding several new results.

2. DOMAIN OF OPERATORS ENLARGED TO INCLUDE FUNCTIONS WITH NONCOMPACT SUPPORT

In many interesting cases in which a sequence of positive linear operators occurs, the domain includes functions with noncompact support. Korovkin's theorem does not apply directly to those, [4, p. 14] and [3, p. 7], though in Korovkin's book [4, pp. 53–54] it is made clear that provided that the functions belonging to the domain are bounded, in some special cases the theory applies. Sikkema [8] treats, in a somewhat different context, the case of functions with non-compact support and of polynomial growth.

In treating unbounded functions with noncompact support, it is important that proper bounds on the functions be required, as is seen from the following example: consider the real functions f in $C[0, \infty)$ for which $\lim_{x\to\infty} f(x)\, e^{-x}$ exists, and let $L_n(f(t); x) = B_n[f(t); x)] + f(n)\, e^{-n}$ (where $B_n[f]$ is Bernstein's polynomial). Then $L_n(t^k, x) \to x^k$ for $k = 0, 1, 2$, uniformly in $[0, 1]$, but $L_n(e^t, x) \to e^x + 1$, uniformly in $[0, 1]$.

In this section modified versions of Korovkin's Theorem and theorems concerning rate of convergence will be given. While there is neither claim of optimality nor of simplicity of statement, these modified versions are very readily applicable to a number of quite important special cases.

We shall need the following definition that will sum up the conditions for our modified approximation theorems:

DEFINITION 2.1. A sequence $L_n(f(t); x)$ of linear-positive operators is of type $\mathscr{K}(T, S, \mu)$ if the domain of each L_n consists of all functions (or all measurable functions) on T satisfying there $|f(t)| \leqslant M(f)(t^2+1)\mu(t)$ and if

(a) $\| L_n(t^k; x) - x^k \|_{C(S)} = o(1)$ as $n \to \infty$, for $k = 0, 1, 2,$

and

(b) $$\| L_n((t-x)^2 \mu(t); x)\|_{C(S)} \leqslant K \| L_n((t-x)^2; x)\|_{C(S)} \equiv K(\mu_n(S))^2 \equiv K\mu_n^2 = o(1),$$

where $T \subset (-\infty, \infty)$ is closed, $S = T \cap [a, b]$, $-\infty < a < b < \infty$, and $\mu(t) \geqslant 1$.

THEOREM 2.1. *Let $L_n(f(t); x)$ be a sequence of linear positive operators of type $\mathscr{K}(T, S, \mu)$.*

(A) *For $f(x)$ continuous on S_1, $S_1 \subset S$, $S_1 = \overline{S_1}$, we have*

$$\| L_n(f(t); x) - f(x)\|_{C(S_1)} \to 0;$$

(B) *If, in addition, $S_1 = [a, b]$ and $S_2 = [a_2, b_2] \subset S_1$, and if, for some $\eta > 0$, $[a_2 - \eta, b_2 + \eta] \cap T \cap \{(-\infty, \infty) - S_1\} = \varnothing$, then*

$$\| L_n(f(t); x) - f(x)\|_{C(S_2)} \leqslant \|f(t)\| \cdot \| L_n(1, x) - 1 \| + \| L_n(1, x) + 1 \| \, \omega(f, \mu_n) + L\mu_n^2. \quad (2.0)$$

(C) *If, in addition, f' is continuous in S_1, then*

$$\| L_n(f(t), x) - f(x)\|_{C(S_2)} \leqslant \| L_n(1; x) - 1 \| \cdot \|f\| + \| L_n(t-x; x)\| \cdot \|f'(x)\| + \omega(f', \delta)(\mu_n^2\delta^{-1} + \mu_n \| L_n(1; x)\|^{1/2}) + L\mu_n^2.$$

Here ω is the modulus of continuity on S_1, the norms are sup on S_2, and L is a constant.

COROLLARY. *If $L_n(1, x) = 1$ and $L_n(t, x) = x$, then the conditions of Theorem 2.1.C. imply*

$$\| L_n(f; x) - f(\cdot)\|_{C(S_2)} \leqslant 2L\mu_n\omega(f', \mu_n) + L\mu_n^2.$$

Proof. We shall prove B following Shisha and Mond. (A) and (C) following Korovkin and DeVore, can be proved similarly. Let $\delta > 0$. For $x \in S_2$, $t \in [a_2 - \eta, b_2 + \eta] \cap T$ (and therefore $t \in S_1$), we have, as in [6] (with $\omega(\cdot) \equiv \omega(f, \cdot)$),

$$|f(t) - f(x)| \leqslant \omega(|t - x|) \leqslant (1 + (t - x)^2 \delta^{-2}) \omega(\delta). \tag{2.1}$$

For $x \in S_2 = [a_2, b_2]$, $t \in \{(-\infty, \infty) - [a_2 - \eta, b_2 + \eta]\} \cap T$, we can write $|f(t)| \leqslant M(f)(t^2 + 1) \mu(t) \leqslant M(f) M_1(t - x)^2 \mu(t)$. Also, for $x \in S_2$ we have $|f(x)| \leqslant \|f\|_{C(S_2)} \eta^{-2}(t - x)^2$. Combining these estimates we can deduce for $t \in T$ and $x \in S_2$,

$$|f(t) - f(x)| \leqslant (1 + (t - x)^2 \delta^{-2}) \omega(\delta) + M(f) M_1(t - x)^2 \mu(t) + \|f\|_{C(S_2)} \eta^{-2}(t - x)^2,$$

from which (2.0) follows easily, taking $\delta = \mu_n$ (If $\mu_n = 0$, one also can obtain (2.0)). An estimate for L in (2.0) is $L \leqslant M(f) M_1 K + \|f\|_{C(S_2)} \eta^{-2}$ where

$$M_1 = \sup_{\substack{|t-x| \geqslant \eta \\ x \in S_2}} \frac{t^2 + 1}{(t - x)^2}.$$

3. Application to Baskakov and Szász Operators

The Baskakov operators are defined by (see [9])

$$M_n(f(t); x) = \sum_{k=0}^{\infty} (-1)^k \frac{\phi_n^{(k)}(x)}{k!} x^k f\left(\frac{k}{n}\right), \tag{3.1}$$

where $\phi_n(x)$ are real functions for which

(a) $\phi_n(0) = 1$;

(b) $(-1)^k \phi_n^{(k)}(x) \geqslant 0$, $k = 0, 1, \ldots$, $x \in [0, \infty)$;

(c) $\phi_n^{(k)}(x) = -n\phi_{n+l}^{(k-1)}(x)$, $k = 1, 2, \ldots$, $x \in [0, \infty)$, for some integer l independent of n, k and x; and

(d) $\phi_n(x)$ can be expanded in a Taylor series in $[0, \infty)$. We shall show $M_n \in \mathscr{K}(T; \mu)$ with $T = [0, \infty)$ and $\mu(t) = (t^l + 1)^{1/2}$, where l is any integer, and a sequence of linear positive operators belongs to $\mathscr{K}(T, \mu)$ if it is of type $\mathscr{K}(T, S, \mu)$ for all appropriate S.

Obviously the M_n are linear positive operators. Suzuki [9, p. 436] showed:

$$M_n\{(t - x)^4; x\} \leqslant K(S)(1/n^2) \qquad \text{for} \quad x \in S, \quad S \text{ compact}. \tag{3.2}$$

Also, using the same technique of reduction [8, pp. 334–336], one obtains for any integer l,

$$M_n\{t^l; x\} \leqslant K_1(S) \quad \text{for} \quad x \in S. \tag{3.3}$$

Therefore for $x \in S$,

$$\begin{aligned} M_n\{(t-x)^2(t^l+1)^{1/2}, x\} &\leqslant [M_n\{(t-x)^4, x\}^{-1/2}\, M_n\{t^l+1, x\}]^{1/2} \\ &\leqslant K_2(1/n). \end{aligned}$$

As corollaries of our theorem we get for $|f(t)| \leqslant K(t^l+1)^{1/2}(t^2+1)$,

$$\| M_n(f; x) - f(x)\|_{C(S)} \to 0, \tag{3.4}$$

where S is compact and $f(t)$ is continuous on S;

$$\| M_n(f; x) - f(x)\|_{C(S_1)} \leqslant 2\omega((b(1+lb)/n)^{1/2}) + L_1(b)\, n^{-1}; \tag{3.5}$$

and

$$\| M_n(f; x) - f(x)\|_{C(S_1)} \leqslant 2(b(1+lb)/n)^{1/2}\, \omega(f', (b(1+lb)/n)^{1/2}) + L_1(b)\, n^{-1}; \tag{3.6}$$

where $S_1 = [a+\eta, b-\eta]$, $\eta > 0$, $S = [a, b]$ and $\omega(\delta)$ is the modulus of continuity of f on S.

The Szász operators defined by

$$S_n(f(t); x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{n^k}{k!} x^k e^{-nx} \tag{3.7}$$

are a special case of the Baskakov operators with $l = 0$. We shall prove here $S_n \in \mathscr{K}(T, \mu)$ where $T = [0, \infty)$ and $\mu(t) = e^{At}$ with any $A \geqslant 0$. Of course $\mu(t) = e^{At}$ tends to infinity much faster than any polynomial used for Baskakov operators. Moreover, $\mu(t) = e^{At}$ is a good estimate since for $f(t) = \exp(t^{1+\epsilon})$, $\epsilon > 0$, S_n is not defined.

We have

$$S_n(e^{At}; x) = \exp(nx(e^{A/n} - 1)),$$

and as $e^{A/n} - 1 \leqslant (A/n)\, e^{A/n}$ for $A \geqslant 0$,

$$S_n((t-x)^2\, e^{At}; x) \leqslant \exp(xAe^{A/n}) \left\{x^2 \left(\frac{A}{n}\right)^2 e^{A/n} + \frac{x}{n}\, e^{A/n}\right\}.$$

Therefore, recalling that $S_n((t-x)^2; x) = x/n$, the proof that $S_n \in \mathscr{K}([0, \infty), e^{At})$ is completed.

Furthermore,

$$\|f(x) - S_n(f(t); x)\|_{C(S_1)} \leqslant 2\omega((b/n)^{1/2}) + L_1(S)\, n^{-1} \tag{3.8}$$

and

$$\|f(x) - S_n(f(t); x)\|_{C(S_1)} \leqslant 2(b/n)^{1/2}\, \omega(f', (b/n)^{1/2}) + L_1(S)\, n^{-1}. \tag{3.9}$$

(3.5)–(3.9) appear to be new results.

4. Further Applications, Convolution Operators

In this section linear positive operators arising from inversion formulae of a wide class of convolution transforms will be discussed. Functions operated on will be defined and measurable on $(-\infty, \infty)$. (In the previous section measurability was not needed.) We shall show that the operators I_n below belong to $\mathscr{K}(T, \mu(t))$ where $T = (-\infty, \infty)$.

The inversion of the Weierstrass transform yields the positive operators

$$I_n(f(t); x) = (n/4\pi)^{1/2} \int_{-\infty}^{\infty} \exp(-(t - x)^2\, n/4)\, f(t)\, dt, \qquad n = 1, 2, 3, \ldots . \tag{9.1}$$

We shall prove $I_n \in \mathscr{K}(T, e^{t^2/4})$. Using [3, p. 78], we have

$$\begin{gathered} I_n(1, x) = 1, \qquad I_n(t; x) = x, \\ I_n(t^2; x) = x^2 + (2/n) \qquad \text{and} \qquad I_n((t - x)^2; x) = 2/n. \end{gathered} \tag{4.2}$$

Straightforward calculations yield that the conditions of Definition 2.1 are satisfied.

Therefore, for $|f(t)| \leqslant M(t^2 + 1)\, e^{t^2/4}$ and S, S_1, S_2, η and $\omega(\cdot)$ as in Theorem 2.1, we have:

$$\|f(x) - I_n(f(t); x)\|_{C(S)} \to 0 \qquad \text{if } f(t) \text{ is continuous on } S; \tag{4.3}$$

$$\|f(x) - I_n(f(t); x)\|_{C(S_2)} \leqslant 2\omega((2/n)^{1/2}) + L_1(a, b, \eta)\, n^{-1}; \tag{4.4}$$

and

$$\|f(x) - I_n(f(t); x)\|_{C(S_2)} \leqslant 2((2/n)^{1/2})\, \omega(f', (2/n)^{1/2}) + L_1(a, b, \eta)\, n^{-1}. \tag{4.5}$$

The linear positive operators I_n induced by Hirschman–Widder's real inversion formula (see [1, Chap. VI]) are given by

$$I_n(f(t); x) = \int_{-\infty}^{\infty} G_n(t - x)\, f(t)\, dt, \qquad n = 0, 1, 2, \ldots, \tag{4.6}$$

where

$$G_n(t) = (1/2\pi i) \int_{-i\infty}^{i\infty} \left(\prod_{k=n+1}^{\infty} (1 - s/a_k)\, e^{s/a_k} \right)^{-1} e^{st}\, ds, \tag{4.7}$$

a_k are real and $\sum a_k^{-2} < \infty$.

We can show $I_n \in \mathscr{K}(T, \exp A \mid t \mid)$. From [2, Chap. VI] we have $I_n(1; x) = 1$, $I_n(t; x) = x$ and

$$I_n((t-x)^2; x) = \sum_{k=n+1}^{\infty} a_k^{-2} \equiv \sigma_n^{\;2}. \tag{4.8}$$

We can write for $n \geqslant n(A)$

$$I_n((t-x)^2 \exp A \mid t \mid, x) \leqslant \{I_n((t-x)^4, x)\}^{1/2} \{I_n(\exp 2\, A \mid t \mid, x)\}^{1/2}.$$

Using Theorem 4.3 of [2, p. 278], we have $I_n((t-x)^4, x) \leqslant k_2 \sigma_n^{\;4}$, and in addition, $I_n(\exp 2\, A \mid t \mid, x)$ is bounded for n sufficiently large and x in a compact set.

Therefore, for $| f(t) | \leqslant M e^{A|t|}$, we have

$$\| I_n(f(t); x) - f(x) \|_{C(S_2)} \leqslant 2\omega(\sigma_n) + L_1(a, b)\, \sigma_n^{\;2}, \tag{4.9}$$

and

$$\| I_n(f(t); x) - f(x) \|_{C(S_2)} \leqslant 2\sigma_n \omega(f', (\sigma_n) + L_1(a, b)\, \sigma_n^{\;2}. \tag{4.10}$$

(4.9) and (4.10) are new results.

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